

Magnetic Field Error Coefficients for Helical Dipoles

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1 Introduction

The aim of this paper is to give a notation for the magnetic field error coefficients of helical dipoles. These coefficients shall be the magnetic multipole coefficients of ordinary dipoles when the helical wave length tends to infinity. Such a notation is different from Ref. [1].

For comparison, the magnetic field error notation for ordinary dipoles will be presented first. The notation for helical dipoles is given thereafter.

2 Magnetic Field Errors of Ordinary Dipoles

In a current free region in vacuum where the electrical field \vec{E} is constant, the magnetic field \vec{B} can be derived from a scalar potential ψ as

$$\vec{B} = -\nabla\psi. \quad (1)$$

We will use a Cartesian coordinate system (x, y, z) and a cylindrical coordinate system (r, θ, z) . Here, x denotes the horizontal, y the vertical and z the longitudinal direction. Furthermore we have

$$\begin{aligned} x &= r \cos \theta, \\ y &= r \sin \theta. \end{aligned} \quad (2)$$

We consider a dipole of infinite length, thus neglecting fringe fields. The symmetry condition of such an element reads

$$\psi(r, \theta, z) = \psi(r, \theta, z + \Delta z) \quad (3)$$

where Δz is arbitrary. Therefore, the potential ψ is independent of z :

$$\psi(r, \theta, z) = \psi(r, \theta). \quad (4)$$

Having a main field B_0 in y -direction, the solution of the Laplace equation $\Delta\psi = 0$ can be written in cylindrical coordinates as

$$\psi(r, \theta) = -B_0 \left\{ r \sin \theta + \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{r^{n+1}}{r_0^n} [a_n \cos((n+1)\theta) + b_n \sin((n+1)\theta)] \right\}. \quad (5)$$

The term $-B_0 r \sin \theta$ gives the main field and the coefficients a_n and b_n denote deviations from the main field. The b_n are called “normal” and the a_n “skew” multipole coefficients. Here, the subscript “0” denotes a dipole, “1” a quadrupole etc. r_0 is a reference radius. For the RHIC dipoles $r_0 = \frac{5}{8} r_{coil}$ is used with $r_{coil} = 40$ mm.

From equations (1) and (5) the magnetic field can be obtained in cylindrical coordinates. We have

$$\begin{aligned} B_r &= B_0 \left\{ \sin \theta + \sum_{n=0}^{\infty} \left(\frac{r}{r_0} \right)^n [a_n \cos((n+1)\theta) + b_n \sin((n+1)\theta)] \right\}, \\ B_\theta &= B_0 \left\{ \cos \theta + \sum_{n=0}^{\infty} \left(\frac{r}{r_0} \right)^n [b_n \cos((n+1)\theta) - a_n \sin((n+1)\theta)] \right\}, \\ B_z &= 0. \end{aligned} \quad (6)$$

The Cartesian components of \vec{B} can be written as

$$\begin{aligned} B_x &= B_0 \left\{ \sum_{n=0}^{\infty} \left(\frac{r}{r_0} \right)^n [a_n \cos(n\theta) + b_n \sin(n\theta)] \right\}, \\ B_y &= B_0 \left\{ 1 + \sum_{n=0}^{\infty} \left(\frac{r}{r_0} \right)^n [b_n \cos(n\theta) - a_n \sin(n\theta)] \right\}, \\ B_z &= 0, \end{aligned} \quad (7)$$

which can also be expressed as

$$B_y + iB_x = B_0 \left[1 + \sum_{n=0}^{\infty} (b_n + ia_n) \left(\frac{x + iy}{r_0} \right)^n \right]. \quad (8)$$

Note that the European notation (see for example Ref. [2]) differs from the American one presented here. The transformation is

$$b_n(\text{American}) = b_{n+1}(\text{European}), \quad (9)$$

$$a_n(\text{American}) = -a_{n+1}(\text{European}). \quad (10)$$

3 Magnetic Field Errors of Helical Dipoles

We consider again a magnet of infinite length, thus neglecting fringe fields. The symmetry condition for a helical dipole is

$$\psi(r, \theta, z) = \psi(r, \theta - k\Delta z, z + \Delta z), \quad (11)$$

where Δz is arbitrary. In other words, $\theta - kz = \text{const.}$ $k = 2\pi/\lambda$ is the wave number and λ the wave length of the helix. k shall have the positive sign for right-handed and the negative sign for left-handed helices. Introducing the new variable

$$\tilde{\theta} = \theta - kz, \quad (12)$$

the symmetry condition (11) leads to a potential ψ which is only dependent on r and $\tilde{\theta}$:

$$\psi(r, \theta, z) = \psi(r, \tilde{\theta}). \quad (13)$$

The tilde shall remind the reader of the fact that $\tilde{\theta}$ in a helix is similar to θ in a ordinary dipole. Using $(r, \tilde{\theta})$ as coordinates and having a transverse helical main Field B_0 a solution of the Laplace equation $\Delta\psi = 0$ is (cf. Eq. (5) and Ref. [1])

$$\begin{aligned} \psi(r, \tilde{\theta}) = & -B_0 \left\{ \frac{2}{k} I_1(kr) \sin \tilde{\theta} + \right. \\ & + \sum_{n=0}^{\infty} \frac{2^{n+1}(n+1)!}{(n+1)^{n+2}} \frac{1}{r_0^n k^{n+1}} I_{n+1}((n+1)kr) \times \\ & \left. \times \left[\tilde{a}_n \cos((n+1)\tilde{\theta}) + \tilde{b}_n \sin((n+1)\tilde{\theta}) \right] \right\} \end{aligned} \quad (14)$$

where I_n are modified Bessel functions. Similar to the ordinary dipole case, the term $-B_0 \frac{2}{k} I_1(kr) \sin \tilde{\theta}$ yields the main field and the coefficients \tilde{b}_n, \tilde{a}_n the deviations thereof. Here, the \tilde{b}_n are called “normal” and the \tilde{a}_n “skew” helical multipole coefficients (with respect to the direction of the main field B_0). The subscript “0” denotes a helical dipole, the subscript “1” a helical quadrupole etc. r_0 is again a reference radius.

The factors in (14) are chosen in such a way as to obtain the potential (5) when the helical wave length tends to infinity. In this case $k \rightarrow 0$, $\tilde{\theta} \rightarrow \theta$ and the Bessel function can be approximated by (cf. Ref. [3])

$$I_n(z) \approx \frac{1}{2^n} \frac{z^n}{n!}. \quad (15)$$

Now, the magnetic field can be computed as (cf. Ref. [1])

$$\begin{aligned}
B_r &= B_0 \left\{ 2I_1'(kr) \sin \tilde{\theta} + \right. \\
&\quad + \sum_{n=0}^{\infty} \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \frac{1}{r_0^n k^n} I_{n+1}'((n+1)kr) \times \\
&\quad \times \left[\tilde{a}_n \cos((n+1)\tilde{\theta}) + \tilde{b}_n \sin((n+1)\tilde{\theta}) \right] \Big\}, \\
B_\theta &= -\frac{1}{kr} B_z, \\
B_z &= -B_0 \left\{ 2I_1(kr) \cos \tilde{\theta} + \right. \\
&\quad + \sum_{n=0}^{\infty} \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \frac{1}{r_0^n k^n} I_{n+1}((n+1)kr) \times \\
&\quad \times \left[\tilde{b}_n \cos((n+1)\tilde{\theta}) - \tilde{a}_n \sin((n+1)\tilde{\theta}) \right] \Big\},
\end{aligned} \tag{16}$$

where I_n' denotes the derivative with respect to the argument of the Bessel function.

Since the Bessel function is nonlinear, the magnetic field of a helical dipole is nonlinear too, even the main field given by B_0 . Close to the magnet axis we have $r \rightarrow 0$ and the field can be approximated by

$$\begin{aligned}
B_x &= -B_0 \sin(kz), \\
B_y &= B_0 \cos(kz), \\
B_z &= -B_0 k [x \cos(kz) + y \sin(kz)],
\end{aligned} \tag{17}$$

i.e. even close to the magnet axis there is a longitudinal field component that will lead to coupling.

References

- [1] V. Ptitsin, "Notes on the helical field", RHIC/AP/41 (1994).
- [2] J. Rossbach and P. Schmüser, "Basic course on accelerator optics", Fifth General Accelerator Course, University of Jyväskylä, Finland, CERN 94-01 (1994).
- [3] M. Abramowitz and I. Stegun, "Handbook of Mathematical Functions", Dover, New York (1972).